PROPERTIES AND COMPARISON OF THE BAYES ESTIMAES OF THE BURR MIXTURE PARAMETERS

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ABSTRACT: In this paper a special case of Burr type-XII distribution is discovered as a transformed version of the Pareto distribution. A finite mixture of the said Burr distribution is proposed to model a heterogeneous population that comprise a finite number of subgroups mixed together in an unknown proportion and with observations that are characterized by one of the one of the finite Burr components provided the data is available on the mixture only. An ordinary type-I right censored sample mixture data is considered. An extensive simulation study is conducted to highlight some interesting properties of the Bayes estimates of the proposed Burr mixture assuming conjugate and uninformative priors. A real life application of the proposed mixture is presented as well.

Key Words: Type I ordinary right censoring, Type IV mixture sample, square error loss function

1 INTRODUCTION

[1] has suggested a number of cumulative distribution functions yielding a wide range of values of skewness and kurtosis to be used to fit almost any given set of unimodal data. [2] presents a nice account of the Burr and related distribution. [3] presented the twelve forms for the cumulative distribution function of Burr distribution. [1], [4], [5], [6] and [7] devoted special attention to one of these forms, called Burr Type-XII, with probability density function given as below. Both c and k are shape parameters.

$$f(x) = k\ c\ x^{c-1}(1+x^c)^{-(k+1)},\ x\ \ge 0,\ k\ >\ 0,\ c\ >0$$

In life testing and reliability we confront with many applications where a population under study is supposed to comprise a finite number of subpopulations mixed together in an unknown proportion. If the observations are assumed to be characterized by one of the corresponding finite members of a family of Burr distribution, the use of the finite mixture Burr distribution becomes inevitable. Mixtures of Burr distribution have not been paid much attention in literature so far. [8] have considered a Burr distribution in terms of Graphical tests. We have discovered a special case of Burr Type-XII distribution as a transformed version of the Pareto distribution and the former has an advantage over the latter in terms of its more realistic support which is defined on positive x-axis.

A finite mixture density function with the k component densities of the proposed Burr distribution (but with unknown parameters, $\sim_i,\ i=1,2,\mathrm{K}\ ,k$) and with unknown mixing weights ($\circledR_i,\ i=1,2,\mathrm{K}\ ,k$) is defined as under.

$$f(x)=\sum_{i=1}^{k}\circledR_i\ \sim_i\ (1+x_{ij})^{-(\sim_i+1)}\ ;\ \ 0\ <\ \circledR_i<\ 1,\ i=1,2,\mathrm{K}\ ,k$$

The corresponding mixture Survival function is given by

$$S(T)=\sum_{i=1}^{k}\circledR_i(1+T)^{-\sim_i}$$

where T is the fixed test termination point used in the ordinary type-I, right censoring. We have discovered the following special case of Burr Type-XII for i[th] component of the mixture.

$$f_i(x)=\sim_i\ (1+x\ )^{-(\sim_i+1)},\ i=1,2,\mathrm{K}\ ,k\ ;$$
$$0<\sim_i<\infty\ ;\ 0\le x<\infty$$

It has support on the positive x-axis and hence seems more suitable to fit lifetime data. It has an interesting relation with the Exponential distribution as well through Burr-Exponential link. This proposed Burr distribution is a transformed version of the following Pareto distribution.

$$f_i(x)=\sim_i\ x^{-(\sim_i+1)},\ i=1,2,\mathrm{K}\ ,k\ ;$$
$$0<\sim_i<\infty\ ;\ 1\le x<\infty$$

As quoted by Soegiarso (1992) [9], four types of mixture samples are considered in real life applications. The Type-IV mixture samples consist of unlabeled observations, of which some are labeled afterwards but rest of them are labeled due to censoring. The real life illustration considers a Type-IV mixture sample.

2. THE POSTERIOR DISTRIBUTIONS ASSUMING THE CONJUGATE PRIOR

The suitable informative conjugate prior to be used in this case is the Gamma prior.

2.1 The Bayes estimators assuming the Gamma Prior

A Type-IV mixture sample of size n units from the Type-I mixture model described above under ordinary type-I right censoring is considered. The likelihood function $L(\boldsymbol{\alpha},\boldsymbol{\pi}|\mathbf{x})$ for the censored data is considered as given in equation (1).

Where $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2,\mathrm{K}\ ,\mathbf{x}_k)$ is data; $\mathbf{x}_i=(x_{i1},\ x_{i2},\mathrm{K}\ ,x_{ir_i}),\ i=1,2,\mathrm{K}\ ,k$

$$L(\boldsymbol{\alpha},\boldsymbol{\pi}|\mathbf{x})\propto\{\sum_{i=1}^{k}\circledR_i\ \ (1+T)^{-\sim_i}\}^{n-r}$$
$$\times\mathop{\mathrm{C}}_{i=1}^{k}\{\prod_{j=1}^{r_i}\circledR_i\ \sim_i\ (1+x_{ij})^{-(\sim_i+1)}\} \tag{1}$$

The likelihood function in (2) can take the following form as well

$$L(\boldsymbol{\alpha},\boldsymbol{\pi}|\mathbf{x})\propto\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^{k}}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}(\prod_{i=1}^{k}\circledR_i^{\,r_i+k_i})$$
$$\times(\prod_{i=1}^{k}\sim_i^{r_i})\exp\left\{-\sum_{i=1}^{k}\sim_i\left\{\sum\ln\ (1+x_{ij})+k_i\ln\ (1+T)\right\}\right\} \tag{2}$$

H_{n-r}^{k} denotes the number of all *k-ary* sequences $(k_1,k_2,\mathrm{K}\ ,k_k)$ of non-negative integers as discussed in [10].

Let $\sim_i\sim Gamma(m_i,s_i),\ i=1,2,\mathrm{K}\ ,k$ and $\boldsymbol{\pi}=(\circledR_1,\circledR_2,\mathrm{K}\ ,\circledR_k):\ Dirichlet(1,1,\mathrm{K}\ ,1)$. That is,

$g(\sim_i) \propto \sim_i^{m_i-1}\exp(-s_i\sim_i),\ i=1,2,\mathrm{K}\ ,k.$ Assuming independence, the joint prior is incorporated with the Likelihood (2) to give the following joint posterior distribution of $\sim_i,\ \circledR_i\ (i=1,2,\mathrm{K}\ k)$ as follows.

$$g_G(\boldsymbol{\alpha},\boldsymbol{\pi}|\mathbf{x}) = \Omega_G^{-1}\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}$$

$$\times\left\{\prod_{i=1}^{k}\circledR_i^{\,r_i+k_i}\right\}\left\{\prod_{i=1}^{k}\sim_1^{r_1+m_1-1}\right\}\exp\left\{-\sum_{i=1}^{k}(\sim_i(A_{ik}+s_i)\right\},$$

$$0<\sim_i<\infty,\ 0<\circledR_i<1,\ i=1,2,\mathrm{K}\ ,k.$$

where
$$\Omega_G = \frac{\displaystyle\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}}{\mathrm{B}(r_1+k_1+1,\mathrm{K}\ ,r_k+k_k+1)\displaystyle\prod_{i=1}^{k}\frac{\Gamma(r_i+m_i)}{(s_i+A_{ik})^{r_i+m_i}}}.$$

The following marginal posterior densities are obtained by integrating out the nuisance parameters.

$$g_{iG}(\sim_i|\mathbf{x}) = \Omega_G^{-1}\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}$$

$$\times\mathrm{B}(r_1+k_1+1,\ r_2+k_2+1,\mathrm{K}\ ,r_k+k_k+1)$$

$$\times\left\{\prod_{i\neq 1}\frac{\Gamma(r_i+m_i)}{(s_i+A_{ik})^{r_i+m_i}}\right\}\sim_i^{r_i+m_i-1}\times\exp\{-\sim_i(A_{ik}+s_i)\},$$

$$0<\sim_i<\infty,\ i=1,2,\mathrm{K}\ ,k$$

Marginal distributions of $\circledR_i,\ i=1,2,\mathrm{K}\ ,k$ can be obtained in the same fashion as well.

Under the squared error loss function, the Bayes estimators are the posterior means of $\sim_i,\ \circledR_i,\ i=1,2,\mathrm{K}\ ,k$ with respect to the respective marginal posterior distributions and are presented below.

$$\hat{\sim}_i = \Omega_U^{-1}\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}\mathrm{B}(r_1+k_1+1,\mathrm{K}\ ,r_k+k_k+1)$$

$$\times\frac{\Gamma(r_i+m_i+1)}{(s_i+A_{ik})^{r_i+m_i+1}}\prod_{i\neq j}\frac{\Gamma(r_i+m_i)}{(s_i+A_{ik})^{r_i+m_i}},\ i=1,2,\mathrm{K}\ ,k$$

$$\hat{\circledR}_i = \Omega_U^{-1}\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}$$

$$\times\mathrm{B}(r_1+k_1+1,\mathrm{K}\ ,\ r_i+k_i+2,\mathrm{K}\ ,r_k+k_k+1)\prod_{j=1}^{k}\frac{\Gamma(r_i+m_i)}{(s_i+A_{ik})^{r_i+m_i}},$$

$$i=1,2,\mathrm{K}\ ,k$$

The expressions for the variances of the Bayes estimators are

$$V(\hat{\sim}_i) = \Omega_U^{-1}\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}\mathrm{B}(r_1+k_1+1,\mathrm{K}\ ,r_k+k_k+1)$$

$$\times\frac{\Gamma(r_i+m_i+2)}{(s_i+A_{ik})^{r_i+m_i+2}}\prod_{i\neq j}\frac{\Gamma(r_i+m_i)}{(s_i+A_{ik})^{r_i+m_i}}-\hat{\sim}_i^2,\ i=1,2,\mathrm{K}\ ,k.$$

$$V(\hat{\circledR}_i) = \Omega_U^{-1}\sum_{k_1,k_2,\mathrm{K}\ ,k_k}^{H_{n-r}^k}\binom{n-r}{k_1,k_2,\mathrm{K}\ ,k_k}$$

$$\times\mathrm{B}(r_1+k_1+1,\mathrm{K}\ ,\ r_i+k_i+3,\mathrm{K}\ ,r_k+k_k+1)$$

$$\times\left\{\prod_{j=1}^{k}\frac{\Gamma(r_j+m_j)}{(s_j+A_{jk})^{r_j+m_j}}\right\}-\hat{\circledR}_i^2,\ i=1,2,\mathrm{K}\ ,k.$$

Here $A_{ik} = \sum_{j=1}^{r_i}\ln\ (1+x_{ij})+k_i\ln\ (1+T),\ i=1,2,\mathrm{K}\ ,k$ **;** $\sum_{i=1}^{k}r_i=r$ and $\sum_{i=1}^{k}k_i=n-r$.

2.2 The Posterior Distributions assuming the uninformative Priors

Uninformative priors works in the state of ignorance about the parameter of interest.

2.2.1 The Bayes estimators assuming the uniform Prior

Let us assume a state of ignorance, that is, are uniformly distributed over $(0,\infty)$. Hence $f_i(\sim_i)=k_i,\ 0<\sim_i<\infty\ ,\ i=1,2,\mathrm{K}\ ,k$ and $\boldsymbol{\pi}=(\circledR_1,\circledR_2,\mathrm{K}\ ,\circledR_k):\ Dirichlet(1,1,\mathrm{K}\ ,1).$ Assuming independence we have an improper joint prior that is proportional to a constant which is incorporated with the likelihood (2) to yield a proper joint posterior distribution. After the similar procedure the Bayes estimators assuming the uniform prior which can be obtained by setting $s_i=0,\ m_i=1$ in the expressions for the Bayes estimators assuming the Gamma prior.

2.2.2 The Bayes estimators assuming the Jeffreys Prior

For the proposed Burr distribution, the Jeffreys priors are $g_i(\sim_i)\propto 1/\sim_i,\ i=1,2,\mathrm{K}\ ,k$, $0<\sim_i<\infty$ and $\boldsymbol{\pi}=(\circledR_1,\circledR_2,\mathrm{K}\ ,\circledR_k):\ Dirichlet(1,1,\mathrm{K}\ ,1).$ Assuming independence, the joint prior is incorporated with the likelihood (2) to have the joint posterior distribution of unknown parameters. The Bayes estimators assuming the jeffreys prior which can be obtained by setting $s_i=0,\ m_i=0$ in the expressions for the Bayes estimators assuming the Gamma prior.

3 A SIMULATION STUDY

An extensive simulations study is conducted to investigate the performance of the Bayes estimators in terms of sample size, censoring rate and various parameter points. Samples of sizes $n=50,\ 100,\ 150,\ 250$ from the two component mixture of Burr distribution with parameters $\sim_1,\ \sim_2$ and $\circledR_1$ such that $(\sim_1,\sim_2)\in\{(0.5,1.5),(3,9)\}$ and $\circledR_1\in\{0.4,0.6\}$. Probabilistic mixing was used to generate the mixture data. The censoring rate in the resulting sample is set approximately to 10% and 20%. The properties and comparison of the Bayes estimates are depicted in Tables 1-2 in terms of sample sizes, mixing proportions and censoring rates.

4 A REAL LIFE EXAMPLE

A mixture data presented in Everitt and Hand (1981), $\mathbf{t}=(t_{11},t_{12},\mathrm{K}\ ,t_{1r_1},t_{21},t_{22},\mathrm{K}\ ,t_{2r_2})$, consist of hours to failure for electronic valves, an indicator valve and for a transmitter valve, both used in aircraft radar sets. The category of the failure is not known until the failure occurs. Inspection of failed units allows the engineers to allocate the failed units to two different subpopulations. The total number of tests carried out was 1003. The transformation $x=\exp(-t)-1$ of an exponential distribution yields the said Burr distribution. The sample characteristics required are also made available easily. For instance, $n\text{-}r=20,\ n=1003,\ r_1=891,\ r_2=92,$

$$r=r_1+r_2=983,\ \sum_{j=1}^{r_1}\ln(1+x_{1j})=\sum_{j=1}^{r_1}t_{1j}=151130 \text{ and}$$

Table 1 Bayes estimates (Jeffreys) of Burr mixture parameters and their standard errors (in parenthesis) with $\sim_1 = 3,\ \sim_2 = 9,\ \text{®}_1 = 0.4,\ 0.6$ and censoring rates, $C = 10\%, 20\%$.

$(\sim_1, \sim_2, \text{®}_1)$	n	10% Censoring		
		$\hat{\sim}_1$	$\hat{\sim}_2$	$\hat{\text{®}}_1$
(3, 9, 0.40)	50	3.37042 (1.20231)	9.06943 (2.15165)	0.394986 (0.0691276)
	100	3.13938 (0.703413)	9.07597 (1.41536)	0.397902(0.0498643)
	150	3.09128 (0.559522)	8.917 (1.16236)	0.395241 (0.0414236)
	250	3.04876 (0.382345)	9.0191 (0.850516)	0.398479 (0.0318075)
(3, 9, 0.60)	50	3.1006 (0.774)	9.22706 (2.52174)	0.588959 (0.0720894)
	100	3.04615 (0.481875)	9.07567 (1.71137)	0.595205(0.0488415)
	150	3.00422 (0.362654)	9.04443 (1.31639)	0.595519(0.0397315)
	250	3.0096 (0.267927)	9.04052(1.01117)	0.598233 (0.0305004)
$(\sim_1, \sim_2, \text{®}_1)$	n	20% Censoring		
		$\hat{\sim}_1$	$\hat{\sim}_2$	$\hat{\text{®}}_1$
(3, 9, 0.40)	50	3.89071 (1.82021)	9.05081(2.49022)	0.378545 (0.0764212)
	100	3.49255 (1.17498)	8.91389 (1.8142)	0.386272(0.0618352)
	150	3.26047 (0.780437)	8.88824 (1.42102)	0.391153 (0.0488964)
	250	3.17515(0.564824)	8.92507 (1.08309)	0.39318 (0.0391872)
(3, 9, 0.60)	50	3.4299 (1.17686)	8.9686 (3.0807)	0.570009(0.075344)
	100	3.14887 (0.660037)	8.86383 (2.00151)	0.588208 (0.0554302)
	150	3.08956 (0.468894)	8.94374 (1.59991)	0.59083(0.0424624)
	250	3.04201 (0.331964)	9.01183 (1.17878)	0.597894 (0.0318113)

$$\sum_{j=1}^{r_2} \ln(1 + x_{2j}) = \sum_{j=1}^{r_2} t_{2j} = 22550$$

Burr mixture parameters $(\sim_1, \sim_2, \text{®}_1)$ are evaluated using estimators derived in Sections 2. The Bayes (Jeffreys) estimates $(\hat{\sim}_{11}, \hat{\sim}_{22})$ after an obvious re-parameterization as evident from the functional form of the component densities of the mixture given in Section 1, are found to be $(\hat{\sim}_{11}, \hat{\sim}_{22}) = (1/\hat{\sim}_1, 1/\hat{\sim}_2) = (179.553, 320.513)$ where $\hat{\sim}_1 = 0.00557,\ \hat{\sim}_2 = 0.00312$ are the Bayes (Jeffreys) estimates of Burr mixture parameters with $SE(\hat{\sim}_1) = 0.000204599,\ SE(\hat{\sim}_2) = 0.000414361$ respectively. The standard errors of the lifetime estimates of the mixture are computed as $SE(\hat{\sim}_{11}) = 6.6064,\ SE(\hat{\sim}_{22}) = 34.6218$ The estimate of the proportion parameter of the Davis mixture is $\hat{\text{®}}_1 = 0.901774$ with $SE(\hat{\text{®}}_1) = 0.00968426$. The proposed estimates presented here are superior to those presented in [11] in terms of Bayesian analysis and information on standard error of the estimates.

5 CONCLUSION

The simulation study highlights some interesting properties of the Bayes estimates. The estimates of the component density parameters are generally over-estimated with some rare exceptions in case of the second component. Also the extent of over-estimation is higher in case of the estimates of the first component density. The estimates of the mixing proportion parameter are under-estimated. The variances of the estimates of all the mixture parameters are reduce as the sample sizes increase. Another remark concerning the variances of the estimates of the component density parameter is that increasing (decreasing) the proportion of a component in the mixture reduces (increases) the variance of the estimate of the corresponding parameter.

It is interesting to note that the size of this over or under-estimation is directly proportional to the amount of the censoring rates and inversely proportional to the sample size. Also the extent of over-estimation is more intensive for larger parameter values. The increase in censoring rate increases the variances of estimates of all the mixture parameters. Furthermore, increasing the sample size reduces the variance of all the estimates without any exception. The increase in proportion of a component in the mixture reduces the variance of the estimate of the corresponding parameter. The Bayes (Uniform), the Bayes (Jeffreys) and the Bayes (Gamma) estimates of the parameter of the first component density are over-estimated but the extent of over-estimation is higher in case of Uniform and the least in case of Gamma. On the other hand, the Bayes (Uniform) estimates of the parameter of the second component density are over-

Table 2: A comparison of the Bayes (Uniform) and Bayes (Jeffreys) estimates of Burr mixture parameters and their standard errors (in parenthesis) with $\sim_1 = 0.5,\ \sim_2 = 1.5,\ ®_1 = 0.25$ and censoring rate, $C = 10\%$.

Prior	$®_1$	n	$\hat{\sim}_1$	$\hat{\sim}_2$	$\hat{®}_1$
U	0.40	50	0.611974(0.250889)	1.56884(0.36821)	0.393967(0.072079)
N		100	0.53933(0.126465)	1.53816(0.247555)	0.396633(0.049935)
I		150	0.5238(0.092298)	1.518(0.198886)	0.398937(0.042411)
F		250	0.514029(0.066073)	1.50749(0.14596)	0.398427(0.030777)
O	0.60	50	0.538668(0.13124)	1.63726(0.447465)	0.588103(0.070883)
R		100	0.513777(0.083804)	1.55149(0.277023)	0.597118(0.050738)
M		150	0.508742(0.061413)	1.537(0.217735)	0.598709(0.037749)
		250	0.505952(0.046685)	1.52397(0.169821)	0.59718(0.029653)
J	0.40	50	0.563245(0.212829)	1.51724(0.350963)	0.394582(0.070558)
E		100	0.520936(0.120153)	1.51118(0.24209)	0.396616(0.049599)
F		150	0.512384(0.089417)	1.4997(0.196023)	0.398852(0.042259)
F		250	0.507501(0.0652)	1.49638(0.144842)	0.39834(0.030742)
R	0.60	50	0.519267(0.12693)	1.5424(0.425452)	0.586989(0.070666)
E		100	0.504399(0.082749)	1.50667(0.271674)	0.596691(0.050753)
Y		150	0.502445(0.060887)	1.5077(0.214871)	0.598474(0.037767)
S		250	0.50212(0.046417)	1.50679(0.168578)	0.597064(0.029662)
	0.40	50	0.549771(0.191977)	1.46817(0.327981)	0.549771(0.191977)
		100	0.51722(0.117482)	1.48681(0.235083)	0.51722(0.117482)
G		150	0.510209(0.088367)	1.48359(0.192396)	0.510209(0.088367)
A		250	0.506287(0.064957)	1.48678(0.143341)	0.506287(0.064957)
M	0.60	50	0.515472(0.124858)	1.4655(0.387584)	0.515472(0.124858)
M		100	0.502549(0.082407)	1.47172(0.261841)	0.502549(0.082407)
A		150	0.501148(0.060726)	1.48489(0.209638)	0.501148(0.060726)
		250	0.549771(0.191977)	1.46817(0.327981)	0.549771(0.191977)

estimated as well, while the Bayes (Jeffreys) estimates are generally over- estimated but with some exceptions and the Bayes (Gamma) estimates are under-estimated. It is interesting to note that the all the three Bayes estimates of the mixing proportion are under-estimated.

The Bayes estimates of the parameters of the component densities with informative (Gamma) prior have the least variances than their Uniform and Jeffreys counterparts while the Bayes (Jeffreys) show smaller variances than Bayes (Uniform). However, the variances of the Bayes (Gamma) estimates of the mixing proportion parameter may not be the least all the times. In other words, the Bayes estimates with informative (Gamma) prior seem to be more efficient than their uninformative counterparts with a few exceptions only in case of the mixing proportion estimates. In the real life example, the proposed estimates are superior in terms of the Bayesian analysis, information on and size of standard error of the estimates.

REFERENCES

[1] Burr, I. W. Cumulative frequency functions. *Annals of Mathematical Statistics.* **13,** 215-232(1942).

[2] Tadikamalla, P. R. A look at the Burr and Related Distributions. *International Statistical Review*, **48**, 337-344(1980).

[3] Johnson, N, Kotz, S. and Balakrishnan, N. Continous Univariate distributions, *John Wiley & Sons, Inc. New York.* (1994)

[4] Burr, I. W. On a general system of distributions, III. The simple range. *Journal of the American Statistical Association.* **63,** 636-643(1968).

[5] Burr, I. W. Parameters for a general system of distributions to match a grid of $\sim_3$ and $\sim_4$. *Communications in Statistics*. **2,** 1-21(1973).

[6] Burr, I. W. and Cislak, P. J. On a general system of distributions: I. Its curve-shaped characteristics; II.

The sample median. *Journal of the American Statistical Association*, **63,** 627-635(1968).

[7] Rodriguez, N. A guide to the Burr type XII distributions. *Biometrika*, **64,** 129-134(1977).

[8] Economou, P.and Caroni, C. Graphical Tests for the Assumption of Gamma and Inverse Gaussian Frailty Distributions. *Lifetime Data Analysis*. **11(4),** 565-582(2005).

[9] Soegiarso, R.W. (1992). A covariate model in finite mixture survival distributions. A PhD Thesis. Department of Epidemiology and Biostatistics, Case Western Reserve University.

[10] Chuan-Chong, C. and Mhee-Meng, K. Principles and Techniques in Combinatorics, *World Scientific Publishing Co. Pvt. Ltd. Singapore*. (1992).

[11] Everitt, B. and Hand, D. J. Finite Mixture Distributions. *Chapman and Hall*. (1981)